General classical solution for dynamics of charges with radiation reaction

2001 J. Phys. A: Math. Gen. 34 473

(http://iopscience.iop.org/0305-4470/34/3/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 473–490 www.iop.org/Journals/ja PII: S0305-4470(01)16520-0

General classical solution for dynamics of charges with radiation reaction

S Danko Bosanac

R Boskovic Institute, Bijenicka c 54, Zagreb 10001, Croatia

E-mail: dbosanac@faust.irb.hr

Received 21 August 2000, in final form 5 December 2000

Abstract

The relativistic dynamics of a charge in the presence of the radiation reaction force is solved in general. It is shown that the solution admits mass conversion of the charge into the energy of the electromagnetic field. The mechanism of annihilation is demonstrated on the example of the radiative capture of the electron by the proton.

PACS numbers: 0350D, 1220, 3130J

1. Introduction

The electromagnetic (EM) field is one of the fundamental forces in nature and its classical aspects are very well understood except in one particular case: unification of two dynamics, one for the EM field and the other for charges. The need for this unification was recognized as a fundamental problem nearly a century ago [1, 2] (for a review on the subject see [3]) and formally it was solved within the model of pointlike charges. However, the resulting equations are of a rather special kind, and their solution was never found for a general case (more about this in the next section). The fact that the unification problem is important cannot be denied: without its solution some of the most important effects in nature cannot be properly understood. For example the energy conservation law is violated without properly taking into account the radiation reaction. Another effect is annihilation (and creation) of charges, in which energy that corresponds to the mass of the charge is interchanged with the energy of the EM field. The prevailing view today, however, is that the unification problem cannot be solved within classical physics; instead it should be solved within a more accurate description of nature, the quantum field theory. Although this attitude may be correct the answer is not satisfactory, because the question that one may rightly ask is why in classical physics unification is not possible. For the radiation reaction the answer is partially known [4, 5], and it was summarized in various forms [6], but in essence it is because the uncertainty principle is not properly taken into account [7–9]. The problem of annihilation of charges is different, because the loss of energy to the radiation is accompanied by the disappearance of the mass. It is because of this that there are not even any crude classical models of the annihilation process, and the standard argument is that classical theory preserves the total number of particles while annihilation (or creation) is obviously a process where this is not the case. Therefore a theory that describes these processes should be inherently non-preserving of the number of particles, and one such is the quantum field theory. If the latter is the only choice then the implication is that the annihilation process has no classical description, but this is only a conjecture and until it is proved there will always be room for doubt that classical theory is a complete failure in this respect.

Before further discussing the unification problems the basic equations for the dynamics of a charge are reviewed. If a charge is acted upon by an external EM force then the relevant set of equations is

$$c^2\, d_\tau \vec{p} = \vec{F} = e\nabla(A^\lambda w_\lambda) - e\, d_\tau \vec{A} \qquad \vec{p} = m\vec{w} = m\, d_\tau \vec{r} \tag{1}$$

where d_s designates the derivative with respect to the variable s, the repeated symbol represents summation

$$A^\lambda w_\lambda = \vec{A}\cdot\vec{w} - A_4 w_4 \qquad w_4 = d_\tau(ct)$$

and τ is the invariant time. The mass of the particle is m, and the symbol w refers to the four-velocity while v refers to the ordinary velocity. In later discussions the kinematical mass will be introduced, which is essentially defined as the ratio between the force and acceleration of the particle, and is given by $m_{\text{kin}} = mw_4$. The force on the particle is represented by the vector $\vec{A}$ and the scalar A_4 potentials, and it can be shown quite readily that (1) is equivalent to

$$c\, d_t \vec{p} = -e\nabla A_4 - \frac{e}{c}\partial_t \vec{A} + \frac{e}{c}\vec{v}\times(\nabla\times\vec{A}) = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$$

which is the well known Lorentz force, where $\vec{v}$ is the velocity of the charge, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic components of the EM field, respectively.

The fourth equation is

$$c^2\, d_\tau p_4 = -\frac{e}{c} w_\lambda \partial_t A^\lambda - e\, d_\tau A_4 \qquad p_4 = mw_4 \tag{2}$$

which ensures that the important relationship

$$p_4^2 - \vec{p}^{\,2} = m^2 \tag{3}$$

is satisfied, where the mass m of the particle is assumed to be constant, i.e.

$$d_t m = 0. \tag{4}$$

From equation (2) one derives the energy conservation law, which is given by

$$d_t(mc^2 w_4 + eA_4) = e\,\partial_t A_4 - \frac{e}{c}\,\vec{v}\cdot\partial_t\vec{A} = 0 \tag{5}$$

where in the last step it was assumed that the EM force is not explicitly time dependent. The total energy of the particle is now

$$K = mc^2 w_4 + eA_4 = mc^2 + (w_4 - 1)mc^2 + eA_4 \tag{6}$$

which is in a form analogous to the non-relativistic expression, except that instead of the kinetic energy it has a term that can be represented as the sum of the energy equivalent mc^2 of the mass of the particle, and the energy that results from its motion (kinetic energy). Being a sum of several terms means that neither of the components of the total energy is necessarily conserved, e.g. kinetic energy may change but that is at the expense of the potential energy. In principle this change may also involve the mass of the particle but that must be at the expense of another component in the total energy, say the potential energy. Because the potential has the property to be arbitrary up to a constant the mass would also be so if it were coupled to the potential, which is nonsense. Therefore if the exchange occurs it should be between the mass and the gradient of the potential (force). Is there evidence for this mechanism? One piece of evidence is the annihilation process; say disappearance of the electron–positron pair when it forms positronium. 'Disappearance' means that the identity of the electron and the positron, epitomized in their mass, is lost but the radiation field is observed (or according to modern views two photons). In this process it is obvious that the energy exchange between the mass and the EM field occurs, and an important question is how this happens. The classical model for this process will be discussed here and for modern views on this matter one should consult the literature on quantum field theory [11].

2. Radiation reaction force

The missing link that unifies the dynamics of the EM field and the dynamics of charges is the radiation reaction force. The need to introduce it is justified relatively easily: if accelerated charge radiates then this fact should be manifested as an additional force on the particle so that the energy balance is correct. The force has been studied for the past hundred years, but without gaining any deeper insight into its impact on the classical dynamics of charges. The problem lies in its form, which in the non-relativistic dynamics is given by [5]

$$\vec{F}_r = \frac{2}{3}\frac{e^2}{c^3}\,\dddot{\vec{r}}\ . \tag{7}$$

Throughout the paper scaled coordinates and time will be used, which are defined by the replacements $\vec{r}\,mc/\hbar \rightarrow \vec{r}$ and $ct\,mc/\hbar \rightarrow t$. In these coordinates the complete non-relativistic dynamics equations for a charge are

$$\ddot{\vec{r}} = -\nabla V + \frac{2\alpha}{3}\,\dddot{\vec{r}} \tag{8}$$

where $\alpha = e^2/(\hbar c)$ is the fine-structure constant, and the potential is given in units of mc^2. For example if the electron–proton Coulomb potential is assumed then $V = -\alpha/r$. Equation (8) is of third order in the time derivative, and this fact prevents its straightforward solution: simple integration produces what is called a 'run-away' solution, because the velocity of the charge increases without bounds, even for a free particle. Various attempts to extract a physically meaningful solution, i.e. the one in which the initial acceleration is chosen so that no 'run-away' solution is obtained, are known. Among them the best known is the formulation of dynamics equations in the integro-differential form [5, 12], but the problem is that advanced forces must be known. Despite the arguments that justify advanced forces, the solution is unlikely to be considered physical. The other way of solving the equations is by the so-called 'back integration' [13, 14]. The problem of finding a physical solution is hampered by the fact that numerical integration of dynamics equations forward in time is very unstable. However, numerical stability is achieved by integrating them backward in time from the moment t to $t = 0$. Final values of the position and the velocity of the charge are varied until the initial conditions are reproduced. The 'back integration' is straightforward but it is not feasible for any trajectory that runs for a long time, therefore it has limited application. In conclusion one can say that classical equations of motion that also include the radiation reaction force have not yet been solved in general.

There is, however, a way of solving equation (8) without any assumptions, but before showing how to do it one important relationship should be derived. By multiplying both sides of (8) with $\dot{\vec{r}}$ one obtains

$$d_t\left[\frac{1}{2}(\dot{\vec{r}})^2 + V - \frac{2\alpha}{3}\dot{\vec{r}}\cdot\ddot{\vec{r}}\right] = -\frac{2\alpha}{3}(\ddot{\vec{r}})^2 + \partial_t V$$

where the identity $d_t V = \dot{\vec{r}}\cdot\nabla V + \partial_t V$ was used. If the potential is time independent then the last term on the right is zero and the quantity

$$K = \frac{1}{2}(\dot{\vec{r}})^2 + V - \frac{2\alpha}{3}(\dot{\vec{r}}\cdot\ddot{\vec{r}})$$

is always decreasing in time, except for the uniform motion. The quantity K is the total energy of the particle, and it has additional term besides the kinetic and the potential energy terms. The meaning of this addition will be discussed later. For the moment, however, it should be noted that it is only non-zero where the force (the gradient of the potential) is not zero.

Equation (8) can be solved by an iteration procedure, where in each step the solution from the obtained equations is closer to the exact one. The great advantage of this procedure, as will be shown, is that each approximation makes physical sense, it is quite accurate and, what is important, preserves the energy balance. One starts by making the first approximation, to assume that the solution of equation (8) is obtained from

$$\ddot{\vec{r}} = \vec{F}(\vec{r})$$

in which case

$$\dddot{\vec{r}} = d_t(\ddot{\vec{r}}) = (\dot{\vec{r}}\cdot\nabla)\vec{F}(\vec{r})$$

hence the improved solution is obtained from

$$\ddot{\vec{r}} = \vec{F}(\vec{r}) + \frac{2\alpha}{3}(\dot{\vec{r}}\cdot\nabla)\vec{F}(\vec{r}) + \mathrm{O}(\alpha^2). \tag{9}$$

Higher-order iterations are derived in appendix A, and their convergence is discussed.

It is important to note that equations (9) give the correct energy conservation law, which is shown by multiplying them by $\dot{\vec{r}}$ and transforming the resulting expression into

$$d_t\left[\frac{1}{2}(\dot{\vec{r}})^2 + V - \frac{2\alpha}{3}\dot{\vec{r}}\cdot\vec{F}(\vec{r})\right] = -\frac{2\alpha}{3}\ddot{\vec{r}}\cdot\vec{F}(\vec{r}) \approx -\frac{2\alpha}{3}\vec{F}\cdot\vec{F}$$

where for simplicity it is assumed that the force is not explicitly time dependent. The law is the same as that based on (8) but $\ddot{\vec{r}}$ is replaced by $\vec{F}$. The error is of the order α^2. Therefore equation (9) is also physically sensible even in the extreme cases of strong interaction, although in this case they may not be sufficiently accurate. However, the iteration procedure in appendix A describes how to improve the accuracy of the solution in a systematic way.

Proper treatment of the processes with the radiation reaction force should be relativistic, and for the pointlike particle the appropriate equations are [15]

$$d_\tau w_\mu = F_\mu + \frac{2\alpha}{3}[d_\tau^2 w_\mu - w_\mu\,(d_\tau w_\lambda\, d_\tau w^\lambda)] \tag{10}$$

where the same scaling was used as in (8). The radiation reaction force contains more terms than required by the straightforward generalization of (7), because they must ensure that a very important identity

$$w_4^2 - \vec{w}^2 = 1 \tag{11}$$

is satisfied. This relationship can be used to work with only the equations for the vector components $\vec{w}$, where w_4 is replaced from (11). It can be shown that equation (10) is

$$d_t\vec{w} = \frac{1}{w_4}\vec{F} + \frac{2\alpha}{3}[d_t(w_4 d_t\vec{w}) - \vec{v}\,w_4^2(d_t\vec{w}\cdot d_t\vec{w} - (\vec{v}\cdot d_t\vec{w})^2)] \qquad w_4 = \sqrt{1-w^2} \tag{12}$$

and they can be solved by iterating $d_t\vec{w}$. In the first iteration one puts

$$\vec{F} = w_4\vec{E} + \vec{w} \times \vec{H}$$

but one point should be noted. The time derivative $d_t\vec{F}$ is

$$d_t\vec{F} = (\vec{v} \cdot d_t\vec{w})\vec{E} + d_t\vec{w} \times \vec{H} + w_4 d_t\vec{E} + \vec{w} \times d_t\vec{H}$$

and the terms $d_t\vec{w}$ should be isolated on the left so that on the right the equations contain time derivatives of the coordinates of maximal order one. When this is done the set of equations for $d_t\vec{w}$ is

$$d_t\vec{w} = \frac{\vec{G} + \frac{2\alpha}{3}\,\vec{v} \times (\vec{E} \times \vec{G}) + \frac{2\alpha}{3}\,\vec{G} \times \vec{H} + \frac{4\alpha^2}{9}(\vec{G} \cdot \vec{H})\vec{H} + \frac{4\alpha^2}{9}\,\vec{v} \times [\vec{H} \times (\vec{G} \times \vec{E})]}{1 - \frac{2\alpha}{3}\vec{E} \cdot \vec{v} + \frac{4\alpha^2}{9}H^2 + \frac{4\alpha^2}{9}\vec{E} \cdot (\vec{v} \times \vec{H}) - \frac{8\alpha^3}{27}(\vec{E} \cdot \vec{H})(\vec{v} \cdot \vec{H})} \tag{13}$$

where

$$\vec{G} = \frac{1}{w_4}\vec{F} + \frac{2\alpha}{3}[w_4(d_t\vec{E} + \vec{v} \times d_t\vec{H}) - \vec{v}\,(F^2 - (\vec{v} \cdot \vec{F})^2)].$$

The equations are quite complicated, but they simplify if it is noted that the solution, in the first iteration, is accurate to only order α. When higher powers are neglected in (13) then the set of equations becomes

$$d_t\vec{w} = \frac{\vec{G} + \frac{2\alpha}{3}\,\vec{v} \times (\vec{E} \times \vec{G}) + \frac{2\alpha}{3}\,\vec{G} \times \vec{H}}{1 - \frac{2\alpha}{3}\vec{E} \cdot \vec{v}}. \tag{14}$$

The dynamics equations (13) and (14) are in the following form: ‘acceleration’ on the left, and the ratio of the force to the term

$$m = 1 - \frac{2\alpha}{3}\,(\vec{v} \cdot d_t\vec{w}) \sim 1 - \frac{2\alpha}{3}\,(\vec{v} \cdot \vec{E}) \tag{15}$$

on the right, which means that (15) can be treated as the mass of the charge. That indeed this is the case is shown by deriving the energy conservation law from (12). By multiplying these equations by $\vec{v}$, and after somewhat lengthy manipulation, the equation

$$d_t\left[\left(1 - \frac{2\alpha}{3}\,(\vec{v} \cdot d_t\vec{w})\right) w_4 + A_4\right] = \partial_t A_4 - \vec{v} \cdot \partial_t\vec{A} - \frac{2\alpha}{3}w_4^2[(d_t\vec{w})^2 - (\vec{v} \cdot d_t\vec{w})^2]$$

is obtained, where the force $\vec{F}$ is from (1). This is the energy conservation law where the total energy of the charge is

$$K = \left[1 - \frac{2\alpha}{3}\,(\vec{v} \cdot d_t\vec{w})\right] w_4 + A_4 \tag{16}$$

which is in the same form as (6), where (15) is the mass of the particle. The important feature of this mass is that it changes during the time when the force acts on the particle, but otherwise it is constant and equal to its original value 1. As mentioned in introduction if the mass of particle changes then that should only occur in the manner just described. This is yet another evidence that (15) can indeed be treated as the mass of the charged particle.

It should be recalled that equations (13) and (14) are approximate forms of the exact dynamics equations (10), and they fail when the mass of the charge (15) deviates considerably from unity. This means that for their accuracy the criterion $\frac{2\alpha}{3}\,|\vec{v} \cdot \vec{E}| \ll 1$ should be satisfied. Therefore the fact that the expression (15) can become zero must be taken with a certain reservation, a possible suggestion being that it is an artefact of the approximation. This is not the case because in appendix A it is shown that the singularity persists in all higher-order approximations; the only thing that changes is its location (as will be discussed later). Prediction of this singularity has some serious consequences for the dynamics of charges.

3. Charge in a constant electric field

The simplest example to test equations (10) is to examine the dynamics of a charge in a constant electric field, directed along the x axis. It is assumed that the charge starts in the space where there is no field, enters the field and then exits it. Without assuming this there is a danger of arriving at erroneous conclusions. The electric field with this property is derived from the potential

$$V(x) = E_0\delta \ln \frac{1 + \mathrm{e}^{\frac{x-L}{\delta}}}{1 + \mathrm{e}^{\frac{x}{\delta}}}$$

where L is the length of the interval within which the field is not zero (it starts at $x = 0$), δ is the rate at which the field changes from its non-zero to its constant value and E_0 is the strength of the field.

Equation (13) in one-dimensional dynamics is

$$d_t^2 x = (1 - v^2)^{3/2} \frac{G}{1 - \frac{2\alpha}{3} E v} \tag{17}$$

where G is

$$G = E + \frac{2\alpha v}{3}(w_4 \partial_x E - E^2).$$

For testing the following parameters were chosen: $\delta = 0.1$, $L = 10$ and $E_0 = 1$ (in the scaled variables), while the initial position of the charge is $x_{\mathrm{in}} = -5$ and its initial velocity $v_{\mathrm{in}} = 0.5$. The prediction is that because the radiation reaction lowers the mass of the charge its velocity v is larger relative to the velocity v_0 when that force is not included in the dynamics. The diagram in figure 1 shows the difference $v - v_0$, which is indeed positive after time $t \sim 10$ when the charge reaches $x = 0$. For $x > L$ (after $t \sim 20$) the velocity of the charge is smaller than v_0 because the net energy release through the radiation could only be at the expense of its kinetic energy (see the inset in figure 1).

Another revealing piece of information about dynamics of the charge is its total energy, which is defined by

$$K = \frac{1}{\sqrt{1 - v^2}}\left(1 - \frac{2\alpha}{3} vE\right) + V = K_0 - \frac{2\alpha}{3\sqrt{1 - v^2}} vE \tag{18}$$

and its time dependence is shown in figure 2 by a solid curve. As expected it is not constant and its steady decrease means that the charge radiates when interacting with the field. The energy component K_0 in (18) is associated with the 'bare energy' of the charge and the remainder with the lowering of the mass of the charge. The time dependence of K_0 is shown in figure 2 by a broken curve, and indicates that in the electric field it increases, thus violating the energy conservation law. The correct energy balance originates from the mass lowering contribution, but as soon as the charge crosses the point where the field disappears this term goes to zero, at the expense of its kinetic energy.

4. Negative charge (electron)–infinitely massive positive charge (proton) scattering

The second example is radiative electron–proton capture (the proton is assumed to be pointlike and fixed at the origin). Relativistic equations of motion that need to be solved are given in (13), and if specified for the Coulomb static potential they are

$$d_t \vec{w} = \frac{\vec{G}}{1 - \frac{2\alpha}{3} \vec{E} \cdot \vec{v}} \tag{19}$$

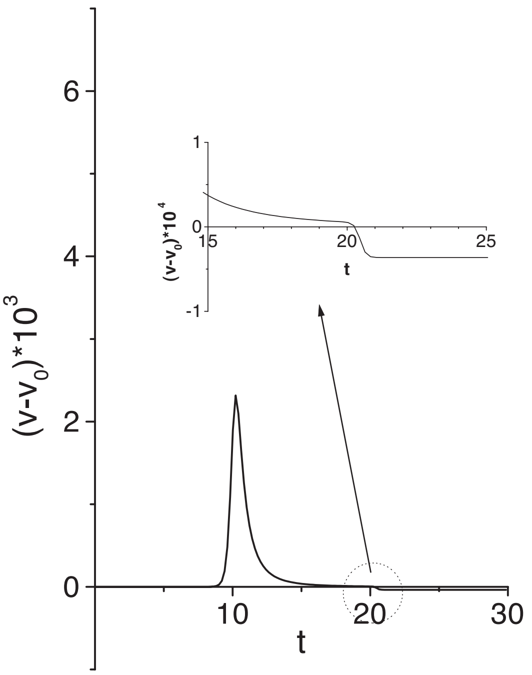

Figure 1. Time dependence of the velocity difference with and without (index 0) radiation reaction included in the dynamics of a charge in a constant electric field. The charge enters the field around time $t = 10$ (it starts in the space where there is no field) and exits it around time $t = 20$. The inset shows the enlarged time interval when the charge exits the field.

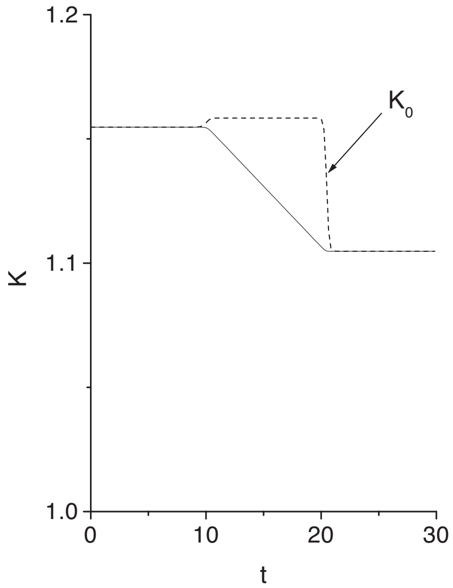

Figure 2. Time dependence of the total energy of the charge from figure 1. The index 0 of the total energy indicates that the mass changing term has been omitted from the total energy.

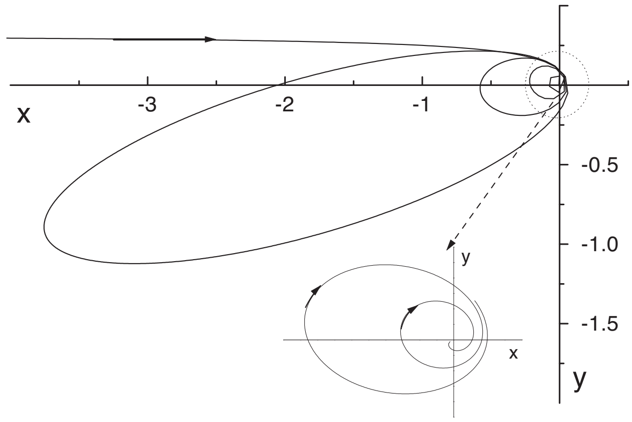

Figure 3. Typical trajectory for the radiative capture of the electron by the proton. In the blown-up segment it is noticed that the trajectory ends before reaching the centre of the proton.

where

$$\vec{G} = \vec{E} + \frac{2\alpha w_4}{3}[(\vec{v}\cdot\nabla)\vec{E} - \vec{v}\, w_4(E^2 - (\vec{v}\cdot\vec{E})^2)]$$

and

$$\vec{E} = \nabla\frac{\alpha}{r} = -\frac{\alpha}{r^3}\vec{r}.$$

In the equations the terms of order α^2 have been neglected because in the first iteration only the first-order terms in the fine-structure constant α are meaningful (see appendix A). A typical numerical solution of these equations is shown in figure 3 for the following initial parameters (the motion is restricted to the x–y plane): the initial position of the electron is $\vec{r}_0 = [-1000, 0.32]$ and its asymptotic initial velocity $\vec{v}_0 = [0.1, 0]$ (again the scaled variables are used). As the electron approaches the proton its energy is lost to the radiation, which results in its capture. The loss of energy continues and the electron spirals into the centre of attraction; however, it never reaches it. A singular point in the numerical integration is encountered, and the trajectory of the charge ends abruptly, as shown in the inset of figure 3. To obtain insight into this process the total energy

$$K = w_4\left(1 - \frac{2\alpha}{3}\,\vec{v}\cdot\vec{E}\right) - \frac{\alpha}{r} = K_0 - \frac{2\alpha}{3}w_4\,\vec{v}\cdot\vec{E}$$

of the electron is calculated, whose time dependence is shown in figure 4. The electron emits radiation in very short, but powerful, bursts of energy whenever it is close to the proton. However, the first is the most powerful because after this the electron is captured. The velocity of the electron is also shown in figure 4, and its rapid change correlates with the bursts of energy.

Much more revealing is the time dependence of the mass of the electron (15), which is shown in figure 5. At each burst of energy its value falls sharply, followed by an increase and then levelling off to its original value as the electron leaves the proton. Towards the end of the trajectory the mass of the electron goes to zero, meaning that the acceleration of the electron goes to infinity, and therefore the singularity is encountered. This means that the electron cannot approach arbitrarily close to the proton because it disappears due to its mass being converted into energy of the radiation field. It can be argued that the electron has lost its identity: in other words it has been annihilated. The distance r_0 from the proton where this

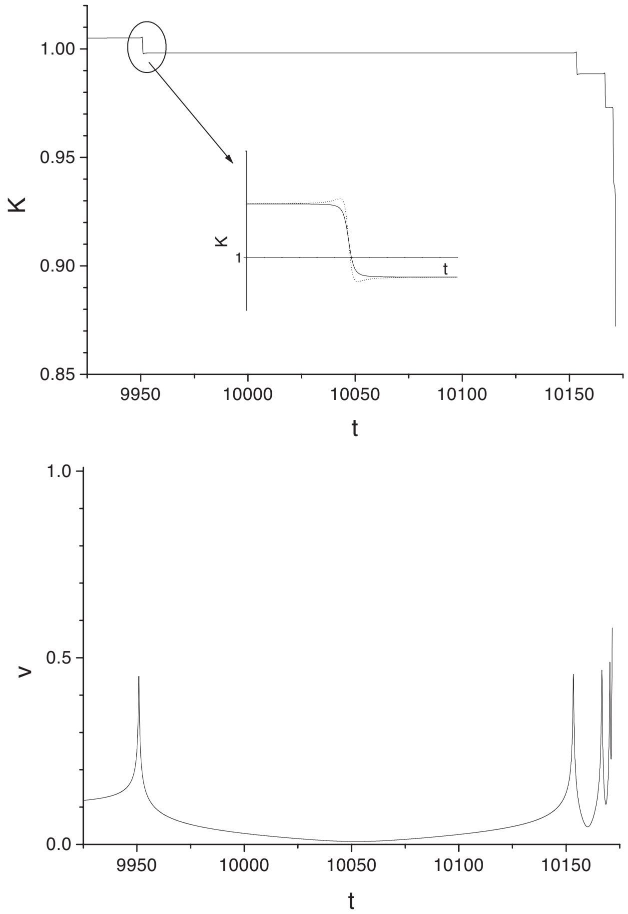

Figure 4. Time dependence of the total energy of the electron from figure 3. In the enlarged time interval within which the first emission of radiation occurs the dotted curve represents the total energy when the mass changing term is not taken into account. The time dependence of the velocity of the electron is also shown. At the end of the trajectory the velocity reaches the speed of light because the mass of the electron goes to zero.

happens can be estimated from the definition of the mass (15). Near the end of the trajectory the electron moves at nearly the speed of light, hence $v \sim 1$, and by putting $m = 1$ one obtains $r_0 = \sqrt{\frac{2}{3}} \frac{e^2}{mc^2}$. This is (apart from the pre-factor) the classical expression that relates the energy equivalent of the mass of the electron to the energy that is stored in its Coulomb field. From figure 5 one estimates the position of the singularity that is in the scaled units $r \sim 0.05$ or $r \sim 1.4 \times 10^{-15}$ m, in agreement with the previous estimate. This estimate is based on the solution of equations (12) in the first iteration. As shown in appendix A, higher-order corrections change this distance by an amount of order $\Delta r \sim 0.02$. It is very important to emphasize that this distance is almost independent of the initial energy of the electron, because most of it is released during the first impact.

Therefore the unique feature of the dynamics of charges, when radiation reaction is included, is that their mass is converted into the energy of the radiated EM field. When all of the mass is converted the trajectory terminates. There are, however, two features that should

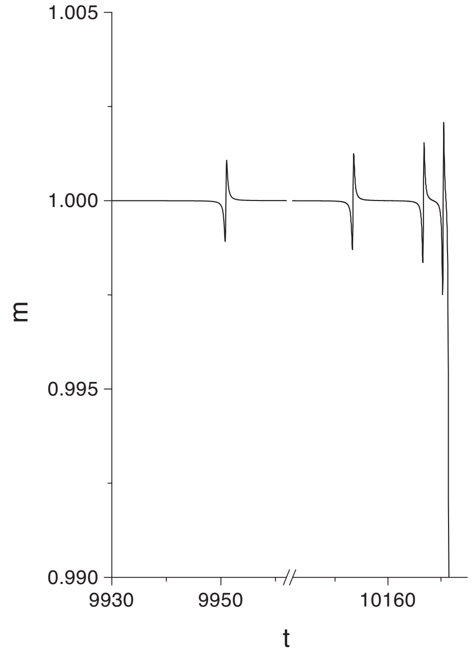

Figure 5. Time dependence of the mass of the electron from figure 3. The rapid changes correlate with the 'fly-by' of the proton by the electron. At the end of the trajectory the mass of the electron goes to zero.

be analysed in a little more detail for the purpose of fully comprehending the consequences of including radiation reaction in the dynamics of charges. One is the time it takes the electron to be annihilated (annihilation time), and the other is the pattern of the radiation field.

The annihilation time of the electron depends on two factors: one is its initial velocity and the other is the impact parameter (deviation from the line of impact for the head-on collision with the proton, normally designated by b). In its definition one must take into account the time t_f it takes the electron to reach the vicinity of the proton, the point of closest approach. If the absolute time when the electron is annihilated is t then the annihilation time is defined as $T = t - t_f$, and for a given initial velocity v of the electron its value is shown in figure 6 as a function of the impact parameter b. The function $T(b)$ has a unique feature that it has a cut-off at large impact parameter, which is relatively simple to explain. At large impact parameters the loss of energy by the electron is too small to be captured, and therefore there must be a value below which this happens. As the impact parameter is lowered below that critical value the annihilation time becomes shorter. Below a certain value of b the process is so fast that it compares with the time for the electron to reach the point of closest approach. In this case the definition of the annihilation time becomes meaningless. Therefore the curves in figure 6 end at some small values of the impact parameter, because annihilation is also very fast.

As the impact parameter is lowered to very small values another effect sets in, which is of considerable importance in the study of relativistic systems. It is shown in appendix B that orbits of relativistic particles around a centre of attraction (no radiation reaction effects are included) show instability of a rather peculiar nature, which has its source in the transversal force. In the non-relativistic dynamics when a particle approaches the centre of attraction (it is assumed that the force on the particle is less singular at the origin than the centrifugal force)

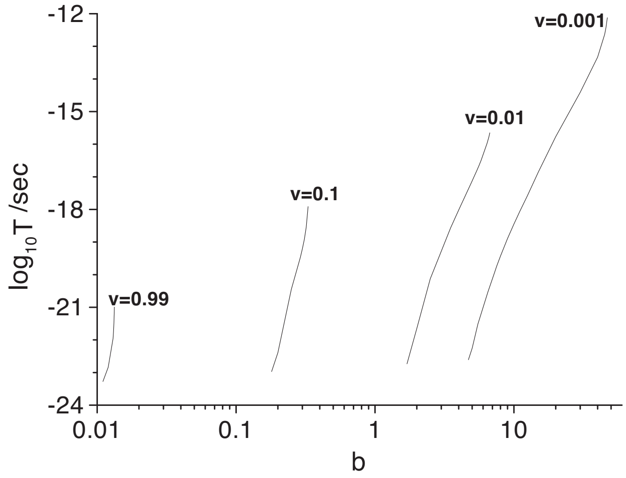

Figure 6. Impact parameter dependence of the annihilation time (in s) for various initial velocities of the electron. The abrupt ending of the curves at their lower points indicates that the definition of the annihilation time becomes inadequate.

at a non-zero impact parameter there is no way it can fall into it. In other words, the particle cannot be 'annihilated' by the centre of attraction. The reason is that at short distances the dominant force is the centrifugal one, which is always repulsive and therefore the particle is repelled from the centre of attraction. In the relativistic dynamics this is also the case, when the transversal force can be neglected. However, this force becomes non-negligible when the orbiting velocity of the particle around the centre of attraction becomes comparable with the speed of light and the attractive force becomes considerable. In this case the effect of the transversal force is to reduce the orbiting velocity at the expense of the radial one, and eventually there is no way the trend can be reversed. As the result the particle falls into the centre of attraction: it is 'annihilated'. The impact parameter when this happens is calculated from the expression (B.1), and when applied to the examples in figure 6 one obtains that the effect sets in at $b \sim 0.001$ for $v = 0.99$, and $b \sim 0.073$ for $v = 0.1$. Therefore for impact parameters smaller than these the annihilation proceeds as a direct process, i.e. no steplike energy release of the kind shown in figure 4 is taking place.

The radiative capture cross section is easily calculated from the data in figure 6. It is, by definition, given by

$$\sigma_{\text{ann}} = \pi b_{\text{max}}^2$$

where b_{max} is the maximum value of the impact parameter for which the annihilation takes place, e.g. for $v = 0.001$ its value is $b_{\text{max}} \sim 47$. The trend of the cross section is to increase as the collision energy decreases; however, more detailed analysis is not made because this is outside the scope of the paper.

Another interesting property of the radiative capture is the structure of the radiated EM field, because it is formed in a very short time interval. For example, if the initial velocity of the electron is $v = 0.99$ then its kinetic energy $E_{\text{kin}} \sim 6.1\, m_{\text{el}}c^2$ is released in a time interval $\Delta t \sim (0.01\text{–}1.3) \times 10^{-23}$ s, while if it is $v = 0.1$ (kinetic energy of the electron is $E_{\text{kin}} \sim 0.005\, m_{\text{el}}c^2$) then this happens in $\Delta t \sim (0.23\text{–}3) \times 10^{-22}$ s. The radiated EM field is therefore in the form of a pulse, which is confined within the radial space interval of the order $\Delta r \sim c\, \Delta t$, and within the solid angle that is determined by the initial velocity of the electron.

The interval of the latter is determined from the radiation pattern of a charge in the relativistic circular motion [5]. The result is that the faster the charge the narrower the solid angle within which the radiated pulse is confined. Both confinement intervals, in the radial direction and the solid angle, stay constant in time, as the EM pulse leaves the collision region. The amplitude of the EM field therefore diminishes as r^{-1}, although in the radial direction the confinement interval stays constant. This means that the first EM pulse is confined to a very small volume in space, thus implying that the strength of the EM field could be extremely large.

The electric component of the EM field that falls off as r^{-1}, which is produced by a charge that undergoes non-uniform motion, is given by

$$\vec{E} = \alpha \left\{ \frac{\hat{n} \times [(\hat{n} - \vec{v}) \times d_t \vec{v}]}{(1 - \hat{n} \cdot \vec{v})^3 R} \right\}_{\mathrm{ret}}$$

where the subscript ret indicates that the quantities in the bracket should be calculated at the retarded time (the scaled coordinates are used). If the field is observed at the position $\vec{\rho}$ and at time t then $R = |\vec{\rho} - \vec{r}(\tau)|$, where $\vec{r}(\tau)$ is the coordinate of the charge at the retarded time τ. The unit vector $\hat{n}$ is in the direction $\vec{\rho} - \vec{r}(\tau)$, and $\vec{v}$ is the velocity of the charge (in the units of c). In order to show how powerful the field that is produced in the radiative capture is its strength is calculated at the distance of the Bohr radius. The Coulomb field at this distance is $E_{\mathrm{Coulomb}} = \alpha/r^2 \sim 4.3 \times 10^{-7}$ while the amplitude of the radiation electric field, if the initial velocity of the electron is $v = 0.99$, is $E \sim 11$. On the other hand if $v = 0.1$ this amplitude is of order $E \sim 10^{-4}$. This example illustrates the dramatic increase in the field strength for very energetic collisions, and the impact it has on the surrounding space is a subject on its own.

5. Summary

Previous analysis was motivated by the long-standing puzzle, at least from the classical point of view, of how two opposite charges annihilate. A little inspection shows that this question is equivalent to asking what the mechanism is by which the mass (the rest mass and not the kinematical mass) of charges could change and eventually vanish. One comment should be made before commencing the search for the answer. No matter what it is, it has only qualitative value if the uncertainty principle is not properly taken into account. Nevertheless, if the solution is found and it is derived from first (classical) principles then it would help in understanding the basic mechanism that drives this process.

It was indeed shown that as the result of unifying the two dynamics there is exchange of energy between the mass and the EM field. This conclusion was reached by solving the dynamics equations in general, and in appendix A it is shown how this is done. Similar to this solution of the classical dynamics equations is that based on the perturbation expansion [10], but there is one big difference between the two. Each equation in the iteration expansion that is described here is a 'stand-alone' one, i.e. it is independent of the other equations. In the perturbation expansion the equations of a certain order are coupled to the equations of lower orders. However, the biggest difference is in predicting singularity in trajectories: the perturbation expansion does not predict this feature. Closer inspection reveals why this is. The difference between the two procedures is the same as expanding a function either in the Taylor series (perturbation expansion) or in the ratio of two polynomials (Pade approximants) (iteration expansion). It is obvious that if a function has a pole then predicting it from the Taylor expansion is difficult while the simplest ratio of two polynomials has this feature incorporated, even in the simplest of approximations. The trajectory of a charge has a singularity, which means that the perturbation expansion fails even before the singular point. On the other hand the iteration expansion carries the information about the possible singularity in its parametrization. This singularity was conveniently explained as the mass change of the particle, and the arguments that support this conjecture were given in previous sections. However, the concept of mass change should be taken with a certain caution: the mass is normally referred to as a constant of the particle, say the rest mass of the electron. It is a number that cannot be changed. What has been referred to as the mass is the quantity (15) that explains the source of singularity in trajectories, but it is surprising that its features are identical to that of the mass.

A subject very closely related to the mass–EM field relationship is the problem of the so-called EM mass. Ever since the discovery of the electron there have been attempts to identify its mass with the energy of the EM field that it produces. It should be pointed out that in this paper this concept is not used: the mass is treated as a parameter that can change its value in the course of interaction. The question that is not answered here is what the origin of the mass of the charges is when they are not interacting with the field.

Appendix A. Solving dynamics with radiation reaction

Non-relativistic equations that include radiation reaction are

$$\ddot{\vec{r}} = -\nabla V + \frac{2\alpha}{3}\dddot{\vec{r}}$$

and they will be solved by the iteration procedure. The zeroth-order solution is

$$\ddot{\vec{r}} = -\nabla V$$

which gives for the third time derivative of the position vector

$$\dddot{\vec{r}} = -\frac{\mathrm{d}}{\mathrm{d}t}\nabla V = -(\dot{\vec{r}}\cdot\nabla)\nabla V - \nabla\frac{\partial}{\partial t}V.$$

For simplicity it will be assumed that the potential V is not time dependent, in which case the new equation with the radiation reaction force is

$$\ddot{\vec{r}} = -\nabla V - \frac{2\alpha}{3}(\dot{\vec{r}}\cdot\nabla)\nabla V.$$

The solution is accurate to order α, and the next higher is obtained by calculating the third time derivative of the position vector as

$$\dddot{\vec{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\left[-\nabla V - \frac{2\alpha}{3}(\dot{\vec{r}}\cdot\nabla)\nabla V\right] = -(\dot{\vec{r}}\cdot\nabla)\nabla V - \frac{2\alpha}{3}(\ddot{\vec{r}}\cdot\nabla)\nabla V - \frac{2\alpha}{3}(\dot{\vec{r}}\cdot\nabla)(\dot{\vec{r}}\cdot\nabla)\nabla V.$$

The equation for dynamics of the charge is now

$$\ddot{\vec{r}} = -\nabla V - \frac{2\alpha}{3}\left[(\dot{\vec{r}}\cdot\nabla)\nabla V + \frac{2\alpha}{3}(\ddot{\vec{r}}\cdot\nabla)\nabla V + \frac{2\alpha}{3}(\dot{\vec{r}}\cdot\nabla)(\dot{\vec{r}}\cdot\nabla)\nabla V\right]$$

but in order to solve it the second time derivatives must be isolated on the left, when the new set of equations is[1]

$$\ddot{\vec{r}} = \vec{G} - \frac{4\alpha^2}{9}\frac{(\vec{G}\cdot\nabla)\nabla V}{1 + \frac{4\alpha^2}{9}\Delta V}$$

where

$$\vec{G} = -\nabla V - \frac{2\alpha}{3}\left[(\dot{\vec{r}}\cdot\nabla)\nabla V + \frac{2\alpha}{3}(\dot{\vec{r}}\cdot\nabla)(\dot{\vec{r}}\cdot\nabla)\nabla V\right].$$

Higher-order corrections are obtained in the same way, but it is obvious that the procedure becomes more complicated.

[1] The author wishes to thank an anonymous referee for noticing a mistake in the original equation.

For the one-dimensional case the iteration procedure simplifies considerably. If the exact equation is

$$\ddot{x} = f(x) + \varepsilon \dddot{x} \tag{A.1}$$

then the first iterated equation is

$$\ddot{x} = f(x) + \varepsilon \dot{x} f'$$

the second

$$\ddot{x} = \frac{1}{1 - \varepsilon^2 f'}(f + \varepsilon \dot{x} f' + \varepsilon^2 \dot{x}^2 f'')$$

and the third

$$\ddot{x} = \frac{[1 - 2\varepsilon^2 f' + \varepsilon^3 \dot{x} f'']f + \varepsilon \dot{x}(f' + \varepsilon^2 \dot{x}^2 f''' - \varepsilon^2 f'^2) + \varepsilon^2 \dot{x}^2 f''}{1 - 3\varepsilon^2 f' - 2\varepsilon^3 \dot{x} f''}$$

where in the last expression terms of order higher than three in ε have been neglected.

Convergence of the iteration procedure is tested on the example of the harmonic force $f(x) = -\omega^2 x$, for which equation (A.1) can be solved exactly. The details of how the exact solution is obtained are not given here, but the general idea is to find the solution of equation (A.1) and adjust the initial acceleration so that the 'run-away' component is not present. This example is quite demanding as a test for the described iteration procedure not only because long-time behaviour must be reproduced (integration over a large number of oscillations) but also the resulting frequency of the harmonic oscillator is different from its original value. Namely the iteration procedure starts from the equations of motion for the original oscillator and it is not clear that the successive iterations would produce a result for that shifted frequency oscillator. For the harmonic force the explicit form of successive equations that solve the problem is

$$\ddot{x}_1 = -\omega^2 x_1 - \varepsilon\, \omega^2 \dot{x}_1$$
$$\ddot{x}_2 = -\frac{1}{1 + \varepsilon^2 \omega^2}(\omega^2 x_2 + \varepsilon\, \omega^2 \dot{x}_2)$$
$$\ddot{x}_3 = -\frac{\omega^2(1 + \varepsilon^2 \omega^2) x_3 + \varepsilon \dot{x}_3 \omega^2}{1 + 2\varepsilon^2 \omega^2}$$

and for testing the specific parameters that were chosen are $\omega = 1$ and $\varepsilon = 0.5$ (relatively large value), while the initial coordinate and the initial velocity are $x = 0$ and $\dot{x} = 1$, respectively; comparison with the exact solution, which is represented by the solid curve, is shown in figure A.1. The successive approximations are represented by the dotted curves, and indicated by arrows. Indeed, iterations converge to the exact solution, and therefore it can be assumed that in the limit of infinite iteration the solution is exact.

The same procedure can be applied to the relativistic equations (12)

$$d_t \vec{w} = \frac{1}{w_4}\vec{F} + \frac{2\alpha}{3}\{d_t(w_4 d_t \vec{w}) - \vec{v}\, w_4^2[d_t \vec{w} \cdot d_t \vec{w} - (\vec{v} \cdot d_t \vec{w})^2]\}$$

which in the first iteration gives

$$d_t \vec{w} = \frac{1}{w_4}\vec{F} + \frac{2\alpha}{3}\{d_t \vec{F} - \vec{v}\,[\vec{F} \cdot \vec{F} - (\vec{v} \cdot \vec{F})^2]\}$$

but the set of equations is not in the final form. It should be recalled that in the force $\vec{F}$ there is a multiplying factor w_4, and its time derivative is proportional to $d_t \vec{w}$, that needs to be isolated on the left. A general solution is (13), which is very complicated, especially if the next correction is sought. For testing the iteration procedure the force is simplified to only the

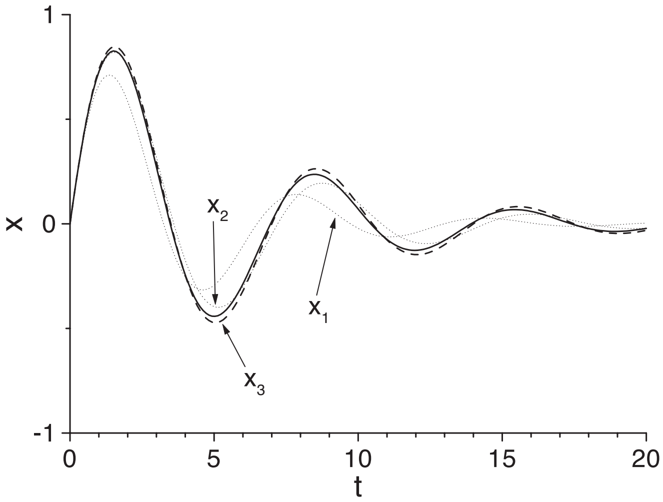

Figure A.1. Convergence of the first three iterations (broken curves) for the trajectory of a charge acted upon by a harmonic force and with the radiation reaction included. The solid curve shows the exact solution.

electric component of the EM field, and the one-dimensional problem is treated. In this case, the first iteration produces the equation

$$d_t w = \frac{G}{1 - \frac{2\alpha}{3} E v} \tag{A.2}$$

where

$$G = E + \frac{2\alpha v}{3}(w_4 \partial_x E - E^2).$$

To obtain the dynamics equations in the next iteration the derivative $d_t w$ from (A.2) is replaced in the equation

$$d_t w = \frac{1}{w_4} F + \frac{2\alpha}{3}[d_t(w_4 d_t w) - v\,(d_t w)^2]$$

but it is obvious that even in this simplified problem the result is complicated. Again, the source of complication is not only w_4, which appears in G, but also velocity in the denominator of (A.2). The details of the derivation are omitted, and only the final result is cited. The dynamics equation for the charge is in the second iteration

$$d_t w = \frac{1}{w_4}\left(1 - \frac{2\alpha}{3} E v\right)\frac{f_n}{f_d}$$

where

$$f_n = \left(1 - \frac{4\alpha^2}{3} w_4 v^2 \partial_x E\right) E - \frac{4\alpha}{3} v E^2 + \frac{4\alpha^2}{9} v^2 E^3 + \frac{2\alpha}{3} v w_4 \left(\partial_x E + \frac{2\alpha}{3} v w_4 \partial_x^2 E\right)$$

and

$$f_d = w_4 \left(1 - \frac{2\alpha}{3} E v\right)^3 - \frac{4\alpha^2}{9} \partial_x E - \frac{8\alpha^2}{9} w_4^2 v^2 \left(1 - \frac{2\alpha}{3} E v\right) \partial_x E.$$

It is obvious that the next iteration is even more complicated.

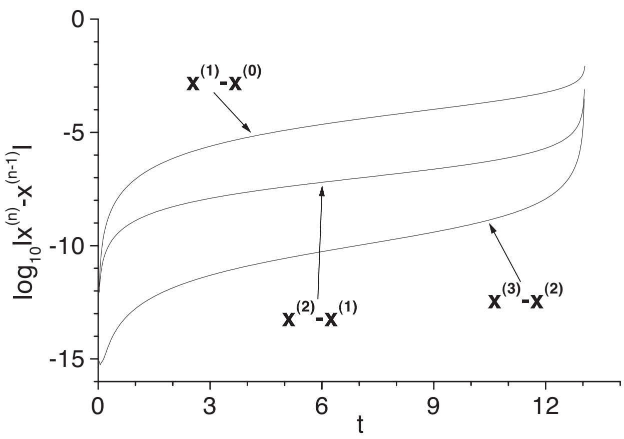

Figure A.2. Convergence of the first three iterations for the trajectory of the electron in the field of the proton and with the radiation reaction included.

Convergence of the iteration procedure is tested on the example of the collinear collision of the electron with the proton. Initially the electron is assumed to be at the distance $x = 1$ and it starts with zero velocity. As it turns out the largest deviation between the first and the second iteration of the exact equation is in this case, and not in the extreme relativistic one, i.e. when the electron has the initial velocity close to the speed of light. However, this statement applies only to the actual trajectory and not the position where it terminates. Namely, the unique feature of the relativistic dynamics with the radiation reaction is that the trajectories encounter singularity because the mass of the charge vanishes in the form of radiation. If the trajectory terminates at x_e then the iterations give $x_e^{(2)} - x_e^{(1)} = 0.009\,61$ and $x_e^{(3)} - x_e^{(2)} = 0.004\,73$, where the superscript indicates the order of iteration (the third iteration was calculated but its analytic value is not given because it is very lengthy). The convergence of the end point is not rapid but there are definite indications that successive iterations give a better value for it. The convergence for the trajectory is demonstrated in figure A.2, where the logarithm of the difference $|x^{(n)} - x^{(n-1)}|$ is shown. As expected, when the radiation reaction effects are small the convergence of the trajectories is rapid; nevertheless, even when they are large (at small electron–proton separations) the convergence is very reasonable. In the extreme relativistic case the first iteration gives a much better solution for the equations (12), but for the end point of the trajectory the corrections of the successive iterations are nearly the same as for the non-relativistic motion.

Appendix B. Stability of relativistic trajectories

If a particle is scattered by centrally symmetric force (the discussion that follows applies to a more general case, but for simplicity only the centrally symmetric force is assumed) then the relativistic equations of motion are

$$d_t \vec{w} = G(r)\,\hat{r} = -\partial_r V(r)\,\hat{r}$$

where the force $G(r)$ has some general functional form. If the motion of the particle is confined to the x–y plane then in the spherical coordinates one constant of motion is the angular momentum

$$p_\phi = w_4 r^2 \dot{\phi}$$

where ϕ is the azimuth angle of the particle. The velocity squared of the particle is

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 = \dot{r}^2 + \frac{p_\phi^2}{r^2}(1 - v^2)$$

and by combining it with the energy conservation law (6) the radial velocity squared is

$$\dot{r}^2 = 1 - \frac{1}{(H - V)^2}\left(1 + \frac{p_\phi^2}{r^2}\right).$$

In the limit $r \to 0$ the centrifugal term dominates the expression if the potential V is not singular, and therefore at one point the radial velocity goes through zero. This means that the particle reaches the point of closest approach, where its radial motion is reversed. By combining the radial velocity with the angular it can be shown that velocity of the particle never approaches the speed of light. The exception is the Coulomb potential, say in the electron–proton system, when for a sufficiently small r the radial velocity squared has the limit (the fine-structure constant α appears because of the scaling that is used in this paper, which was defined in (8))

$$\dot{r}^2 \sim 1 - \frac{p_\phi^2}{\alpha^2}$$

which suggests the existence of two kind of orbit, the dividing line between them being

$$p_\phi = \alpha. \tag{B.1}$$

If $p_\phi > \alpha$ there is again a point of closest approach, and this solution is called stable (for a bound electron the orbit remains elliptic-like indefinitely). However, if $p_\phi < \alpha$ there is no such point and the electron falls onto the proton, and the solution is called unstable (the electron cannot have a stable bound orbit because it always spirals down to the proton). In this case the radial velocity of the electron is smaller than the speed of light, but its velocity reaches it when approaching the centre of the proton. The momentum of the electron becomes infinite. It can be shown that this effect has its origin in what is known as the transversal force. In relativistic dynamics if a force acts on a particle then the perpendicular component of the velocity to the line of force also changes, thus the law of addition of forces from non-relativistic dynamics does not apply. Therefore if a relativistic particle is instantaneously in a circular orbit around the centre of attraction then additional force will appear in the radial direction (transversal force), that always pulls the particle towards it. Sometimes this force becomes larger than the centrifugal one, in which case the particle falls into the centre of attraction.

Previous findingsindicates that the electron can be captured by the proton even without radiation being emitted (the physics of this process can only be meaningfully understood if radiation reaction is included). This happens for impact parameters that are smaller than that determined by (B.1). One can therefore define the radiationless electron capture cross section as[2] $\sigma_{\text{capt}} = \pi b_0^2$, where b_0 is calculated from $p_\phi = \alpha\hbar = m w_4 b_0 v$ (non-scaled coordinates are used again). It follows that

$$\sigma_{\text{capt}} = \pi b_0^2 = \pi r_{\text{cl}}^2\left(\frac{c^2}{v^2} - 1\right)$$

where r_{cl} is the classical radius of the electron.

The effect is manifested in the relativistic (Dirac or Klein–Gordon) quantum treatment of the hydrogen-like atom (the atom in which the proton is replaced by a nucleus of charge Ze). It is known that these equations do not have a solution for those bound states for which

[2] The author wishes to thank an anonymous referee for suggesting inclusion of this estimate.

their total angular momentum J is smaller than $Z\alpha$, which is the same restriction as (B.1). Formally the equations have solution for these J, but not a bound state one, with an interesting property that the wavefunction is rapidly oscillating for small r. The rate of oscillations goes to infinity for $r \to 0$, suggesting that the momentum of the electron becomes infinite in the limit $r \to 0$, the same observation as in classical dynamics. Therefore, the non-existence of relativistic bound state solutions in quantum dynamics is a manifestation of the instability of the orbits.

References

[1] Heaviside O 1902 *Nature* **67** 6
[2] Abraham M 1903 *Ann. Phys., Lpz.* **10** 156
[3] Erber T 1961 *Fortschr. Phys.* **9** 343
[4] Rohrlich F 1997 *Am. J. Phys.* **65** 1051
[5] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)
[6] Barut A O (ed) 1980 *Foundations of Radiation Theory and Quantum Electrodynamics* (New York: Plenum)
[7] Bosanac S D 1994 *Phys. Rev.* A **50** 2899
[8] Doslic N and Bosanac S D 1995 *Phys. Rev.* A **51** 3485
[9] Bosanac S D 1994 *Fizika* B **3** 21
[10] Landau L D and Lifshitz E M 1971 *The Classical Theory of Fields* (New York: Pergamon)
[11] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
[12] Plass G N 1962 *Rev. Mod. Phys.* **33** 37
[13] Huschilt J and Baylis W E 1976 *Phys. Rev.* D **13** 3256
[14] Comay E 1996 *J. Phys. A: Math. Gen.* **29** 2111
[15] Barut A O 1964 *Electrodynamics and Classical Theory of Fields and Particles* (London: MacMillan)